Building a More Profitable Portfolio:

Catastrophe Risk Pricing

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Climbing the Yellow-Stork Tower

The setting sun is disappearing behind the hills,

The once raging yellow river is seaward still.

Come, follow me to the next story,

For breathtaking views await, to show their full glory.

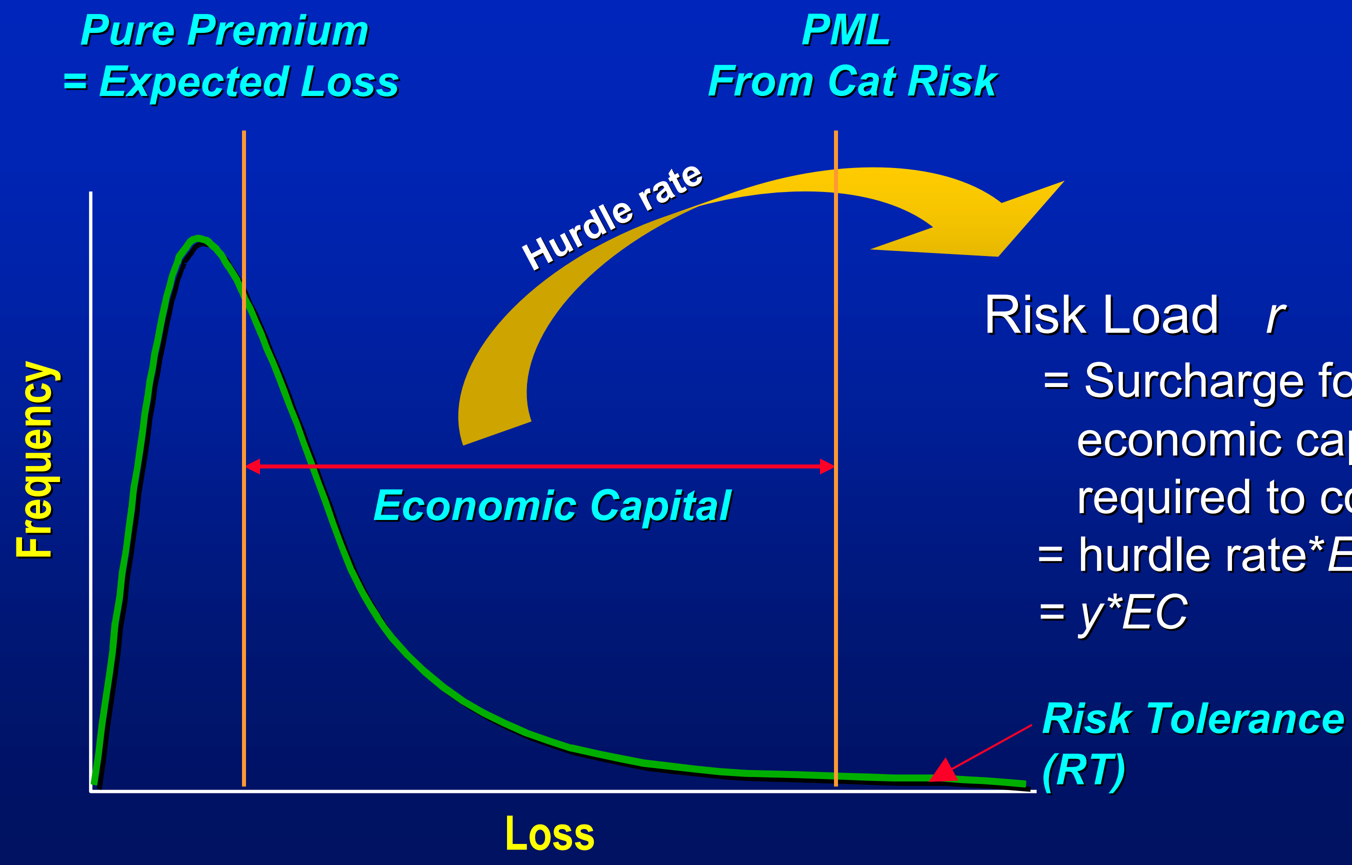
Wang, Zhihuan (688-742) – translation by Felix Wong

Climbing the CAT-Risk Tower

- Single-policy risk load
  - Economic capital, hurdle rate
  - Use of RiskLink® statistics
- Portfolio risk load
  - Diversification benefits, risk correlation
  - Marginal risk load
  - Contract addition order; fair (Shapley) value
  - Renewal, addition, and attrition
  - Portfolio with many small contracts
- CAT risk as part of the enterprise profile
  - Aggregate risk load
  - Diversification factor

CAT Risk Pricing – Single Policy

Total Premium = Pure Premium + Expenses + Risk Load



Risk Load *r*

= Surcharge for additional economic capital (*EC*) required to cover risk

= hurdle rate**EC*

= *y*EC*

CAT Risk Pricing – Single Policy

Event Loss Table			
Event ID	Rate	Mean Loss ($)	CV of Loss
1	λ_1	$\overline{L}_1$	CV_{L_1}
2	λ_2	$\overline{L}_2$	CV_{L_2}
:	:	:	:
j	λ_j	$\overline{L}_j$	CV_{L_j}
:	:	:	:
J	λ_J	$\overline{L}_J$	CV_{L_J}

***EC* = *PML* – Expected Loss**

For example, from RiskLink® outputs:

Expected Loss = μ = $\sum_i \lambda_i \overline{L}_i$

PML = $\mu + z\sigma$

$\sigma = \sqrt{\sum_i \left\{ \overline{L}_i^2 (1 + CV_{L_i}^2) \lambda_i \right\}}$

z = parameter, depends on loss distribution and RT

Hence, *EC* = $(\mu + z\sigma) - \mu = z\sigma$

Therefore, Risk Load = $y \cdot z \cdot \sigma \equiv k\sigma$

CAT Risk Pricing – Single Policy

- Example
  - If loss follows normal (Gaussian) distribution with mean μ and standard deviation σ
  - If PML corresponds to Risk Tolerance (RT) of less than 1% (99 percentile),

    $$z = 2.32 \Rightarrow EC = 2.32\sigma$$

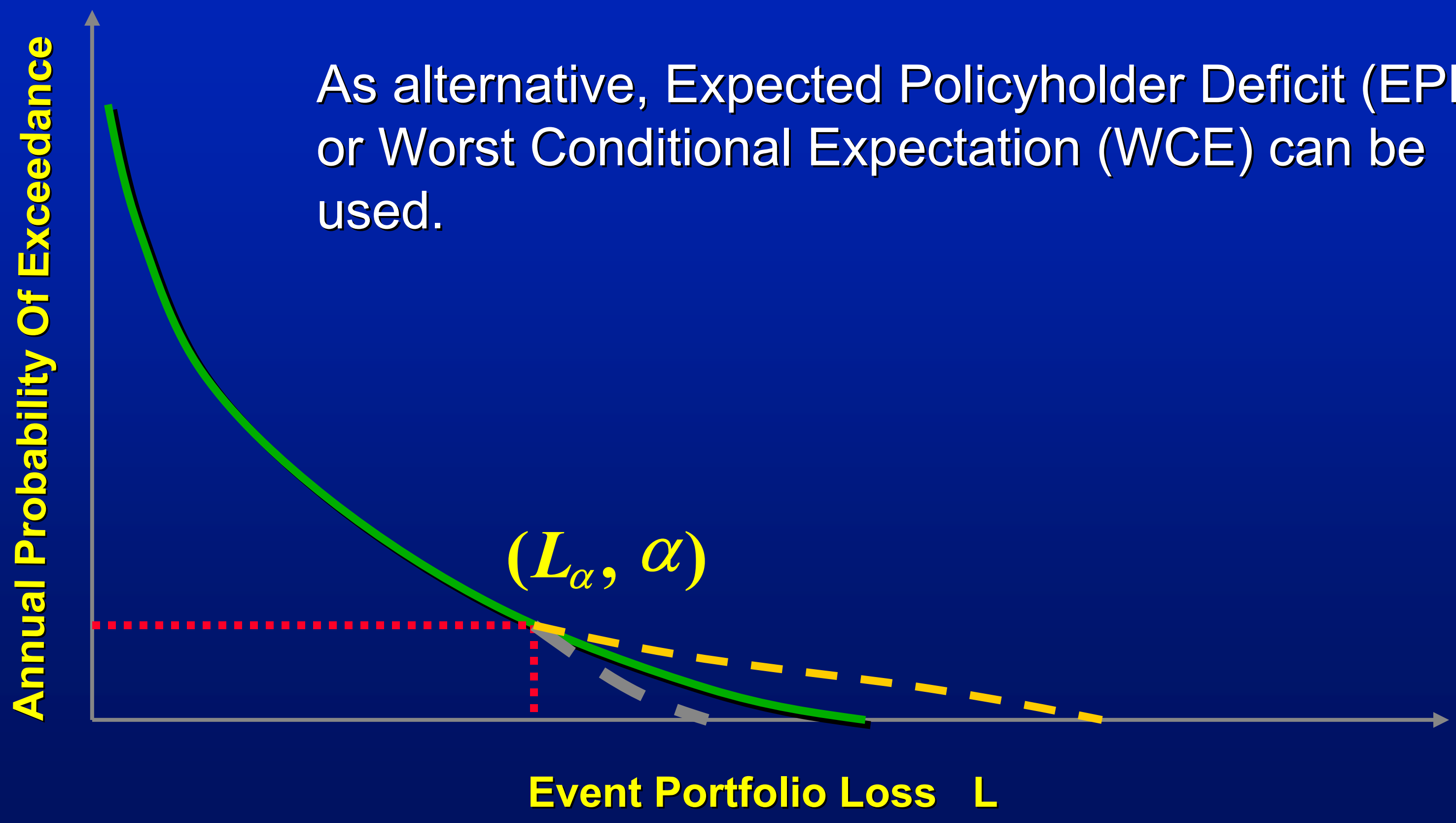
    Or, If PML corresponds to Risk Tolerance (RT) of less than 0.03% (99.97 percentile),

    $$z = 3.43 \Rightarrow EC = 3.43\sigma$$

Tail Independence of Quantile PML

A quantile PML does not distinguish alternative types of tail behaviour.

As alternative, Expected Policyholder Deficit (EPD) or Worst Conditional Expectation (WCE) can be used.



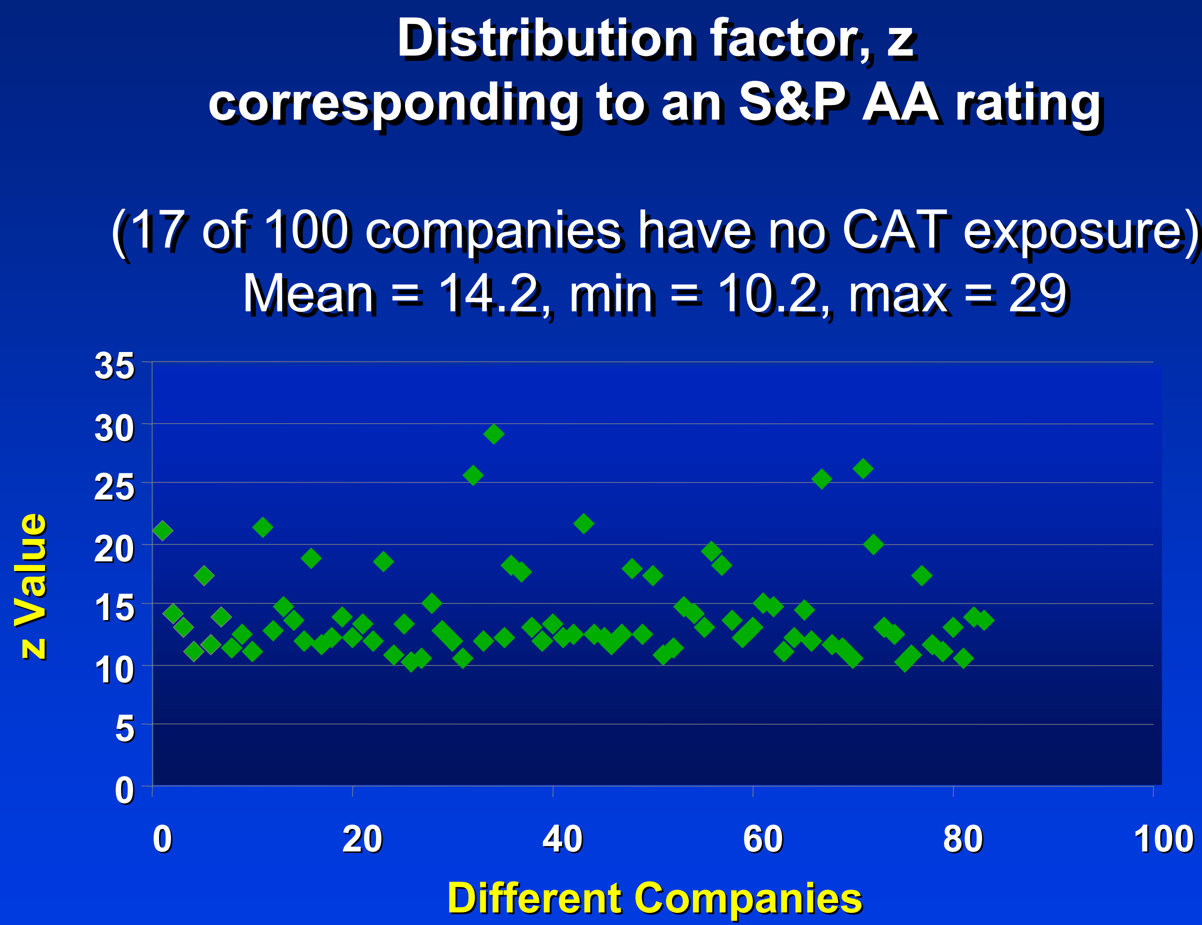
CAT Risk Pricing – Single Policy

Q: Real-world loss distribution is not Gaussian, but highly skewed. Hence, z is not a constant. How should risk load be determined?

A: *z, The Distribution Factor*, depends on an individual company's loss distribution and risk tolerance (RT).

In order to quantify the Distribution Factor, z, the top 100 P&C companies' data were used.

CAT Risk Pricing – Single Policy



Wide range of *z* suggests using company-specific data when available.

CAT Risk Pricing – Single Policy

Q: Aside from shape of distribution, z also depends on risk tolerance *(RT).* What standards should a company use for *RT?*

A: Suggest pegging *RT* to company's rating (S&P, A.M. Best):

"CAT risk should at least not lower the company's current rating."

For instance, *z* in previous chart is based on S&P AA rating (*RT* = 0.03%).

z-charts for other ratings, e.g., A.M. Best A++/A+ (*RT* = 0.02%) are available.

Risk Tolerance Can Be Tied Up with the Company's Rating

S&P Long-Term Credit Ratings versus A.M. Best Financial Strength Ratings

S&P Rating	Probability of Default	A.M. Best Rating	Probability of Default
AAA AA+ AA AA- A+ A	0.01% 0.02% 0.03% 0.04% 0.05% 0.07%	A++/A+	0.02%
A- BBB+ BBB BBB-	0.09% 0.13% 0.22% 0.39%	A/A-	0.21%
BB+ BB BB- B+	0.67% 1.17% 2.03% 3.51%	B++/B+	0.50%
B CCC	6.08% 18.27%	B/B-	No Data

Source: ERisk, A.M. Best, "About Our Ratings", Standard and Poors

CAT Risk Pricing – Portfolios (Multiple Policies)

- Now extend pricing approach to portfolios
  - Portfolios are a business necessity
  - Portfolios provide diversification, lowering risk, and risk load (cost to clients)
- But diversification benefit is not automatic
  - Correlated risks: little diversification
  - Independent risks: full diversification
- Hence, must assemble portfolio judiciously
  - See following examples…

CAT Risk Pricing – Portfolios

- **Example A:** Correlated Risks

Event	λ_i	Policy 1			Policy 2			Portfolio = Policy 1+2		
		L_i	$\lambda_i L_i$	$\lambda_i L_i^2$	L_i	$\lambda_i L_i$	$\lambda_i L_i^2$	L_i	$\lambda_i L_i$	$\lambda_i L_i^2$
1	0.01	100	1	100	10	0.1	1	110	1.10	121
2	0.02	50	1	50	5	0.1	0.5	55	1.10	60.5
3	0.03	30	0.9	27	3	0.09	0.27	33	0.99	32.67
4	0.04	10	0.4	4	1	0.04	0.04	11	0.44	4.84
		$\sigma^2_1=181;\ \sigma_1=13.454$			$\sigma^2_2=1.81;\ \sigma_2 = 1.345$			$\sigma^2_{1+2}=219;\ \sigma_{1+2}=14.799$		

Losses of Policy 1 and 2 are correlated. Marginal risk load for adding Policy 2 to portfolio is 14.799 - 13.454 = 1.345*k*; no gain.

CAT Risk Pricing – Portfolios

- **Example B:** Partly-Correlated Risks

Event	λ_i	Policy 1			Policy 2			Portfolio = Policy 1+2		
		L_i	$\lambda_i L_i$	$\lambda_i L_i^2$	L_i	$\lambda_i L_i$	$\lambda_i L_i^2$	L_i	$\lambda_i L_i$	$\lambda_i L_i^2$
1	0.01	100	1	100	0.6	0.006	0.004	100.6	1.006	101.3
2	0.02	50	1	50	1.8	0.036	0.066	51.8	1.036	53.7
3	0.03	30	0.9	27	3.0	0.09	0.275	33.0	0.99	32.7
4	0.04	10	0.4	4	6.1	0.244	1.466	16.1	0.644	10.3
		σ^2_1=181; σ_1=13.454			σ^2_2=1.81; σ_2 = 1.345			σ^2_{1+2}=198; σ_{1+2}=14.07		

Losses of Policy 1 and 2 are partly-correlated. Marginal risk load for adding Policy 2 to portfolio is 14.07-13.454=0.62*k*; 54% reduction!

CAT Risk Pricing – Portfolios

$$Policy\ \ Sigmas: \quad \sigma_1 = \sqrt{\sum_i \lambda_i \overline{L}_{i,1}^2}; \qquad \sigma_2 = \sqrt{\sum_i \lambda_i \overline{L}_{i,2}^2}$$

$$Portfolio\ \ Sigma: \quad \sigma = \sqrt{\sum_i \lambda_i \overline{L}_i^2}; \qquad \overline{L}_i = \overline{L}_{i,1} + \overline{L}_{i,2}$$

$$Correlation\ Between\ Policies: \quad \rho = \frac{\sigma^2 - \sigma_1^2 - \sigma_2^2}{2\sigma_1\sigma_2}$$

- It can be shown that in Example A, correlation coefficient between Policy 1 and Policy 2 is:

  ρ = 1.0 — Losses are totally correlated; No diversification benefit.

- Whereas In Example B,

  ρ = 0.6 — Losses are partially correlated; Some diversification benefit.

CAT Risk Pricing – Portfolios

- Denote existing portfolio risk by Σ and risk being added by σ:

$$\begin{bmatrix} \Sigma^2 & \rho\sigma\Sigma \\ \rho\sigma\Sigma & \sigma^2 \end{bmatrix}$$

Σ^2 = variance of existing porfolio

σ^2 = variance of additional policy

ρ = correlation coefficient

- Then new portfolio risk is:

$$\Sigma_{new}^2 = \Sigma^2 + 2\rho\sigma\Sigma + \sigma^2$$

- And *marginal* risk load for policy being added is:

$$r = k\Delta\sigma = k(\Sigma_{new} - \Sigma)$$

CAT Risk Pricing – Portfolios

- **Example:** two identical and independent risks

$$\begin{bmatrix} \Sigma^2 & 2\rho\sigma\Sigma \\ 2\rho\sigma\Sigma & \sigma^2 \end{bmatrix} \equiv \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$$

That is:

$$\Sigma = \sigma = 10; \ \ \rho = 0$$

Hence,

$$\Sigma_{new}^2 = 10^2 + 2(0)(10)(10) + 10^2 = 200$$

$$\Sigma_{new} = 14.1$$

$$r = k(\Sigma_{new} - \Sigma) = (14.1 - 10)k = 4.1k$$

$$\text{versus } 10k \text{ if stand - alone}$$

CAT Risk Pricing – Portfolios

Q: But does this mean risk load depends on order that contracts are added to portfolio?

A: YES. In this simple pricing approach, all diversification benefit goes to latest policy added.

Q: Is this fair?

A: NO.

Q: How can we make pricing fair?

A: Seek fair value by taking *ALL* possible addition orders into consideration, and taking the average risk load – the Shapley Value.

CAT Risk Pricing – The Shapley Value

- **Example:** Three contracts C1, C2 and C3 are added to portfolio in that order

$$\begin{matrix} C_1 \\ C_2 \\ C_3 \end{matrix} \quad \begin{bmatrix} 100 & 20 & 30 \\ 20 & 100 & 90 \\ 30 & 90 & 100 \end{bmatrix} \quad \begin{matrix} \sigma_1 = 10 \\ \sigma_2 = 10 \\ \sigma_3 = 10 \end{matrix}$$

Then

$$r_1 = k\Delta\sigma_1 = 10k$$

$$r_2 = k\Delta\sigma_2 = k(\sigma_{1,2} - \sigma_1) = k(\sqrt{240} - \sqrt{100}) = 5.5k$$

$$r_3 = k\Delta\sigma_3 = k(\sigma_{1,2,3} - \sigma_{1,2}) = k(\sqrt{580} - \sqrt{240})k = 8.6k$$

CAT Risk Pricing – The Shapley Value

- When addition order is changed, the contract risk loads will be different

Order of Addition	Individual Contract Risk Load r_i			Portfolio Risk Load
	C_1	C_2	C_3	
$\{C_1, C_2, C_3\}$	10*k*	5.5*k*	8.6*k*	$\sum_i r_i$ = 24.1*k*
$\{C_1, C_3, C_2\}$	10*k*	8*k*	6.1*k*	
$\{C_2, C_1, C_3\}$	5.5*k*	10*k*	8.6*k*	
$\{C_2, C_3, C_1\}$	4.6*k*	10*k*	9.5*k*	
$\{C_3, C_1, C_2\}$	6.1*k*	8*k*	10*k*	
$\{C_3, C_2, C_1\}$	4.6*k*	9.5*k*	10*k*	
Average	6.8*k*	8.5*k*	8.8*k*	

"Fair" Shapley Values

CAT Risk Pricing – The Shapley Value

- The Shapley value can be interpreted as the mathematical *expectation* of the admission value, when all orders of formation of the grand coalition are *equiprobable*.

- In computing the value, one can assume, for convenience, that all players enter the grand coalition one by one, each of them receiving the entire benefit (s)he brings to the coalition formed just before him.

Jean Lemaire, "Cooperative Game Theory and its Insurance Applications", ASTIN Bulletin, Vol. 21, No. 1, 1991.

CAT Risk Pricing – The Shapley Value

- In general form, the Shapley Risk Load for Contract *i* in a portfolio of *n* contracts is:

$$r_i = \frac{1}{n!} \sum_{S \in P(N), i \in S} (s-1)!(n-s)![R(S) - R(S \setminus i)], \qquad i = 1, \dots, n$$

$$N = \text{set of all contracts}$$

$$P(N) = \text{power set of } N$$

$$S = \text{any set of } s \text{ contracts including contract } i$$

$$R(S) = \text{total risk load for contracts in } S$$

$$R(S \setminus i) = \text{risk load for group of contracts w/o contract } i$$

See Following Example

CAT Risk Pricing – The Shapley Value

$$r_i = \frac{1}{n!} \sum_{S \in P(N), i \in S} (s-1)!(n-s)![R(S) - R(S \setminus i)], \qquad i = 1, \ldots, n$$

$$N = [C_1, C_2, C_3], \qquad n = 3$$

$$P(N) = \{\Phi,\ [C_1],\ [C_2],\ [C_3],\ [C_1, C_2],\ [C_2, C_3],\ [C_1, C_3],\ [C_1, C_2, C_3]\}$$

Sets containing contract C_3:

$$S = [C_3], \quad s = 1, \quad R(S) = \sigma_3 = 10, \quad R(S \setminus 3) = 0$$

$$S = [C_1, C_3], \quad s = 2, \quad R(S) = \sigma_{1,3} = \sqrt{260}, \quad R(S \setminus 3) = \sigma_1 = 10$$

$$S = [C_2, C_3], \quad s = 2, \quad R(S) = \sigma_{2,3} = \sqrt{380}, \quad R(S \setminus 3) = \sigma_2 = 10$$

$$S = [C_1, C_2, C_3], \quad s = 3, \quad R(S) = \sigma_{1,2,3} = \sqrt{580}, \quad R(S \setminus 3) = \sigma_{1,2} = \sqrt{240}$$

$$r_3 = \frac{1}{3!}\{2(10-0) + 1 \cdot (\sqrt{260} - 10) + 1 \cdot (\sqrt{380} - 10) + 2 \cdot (\sqrt{580} - \sqrt{240})\}k$$

$$= \frac{1}{6}\{2 \cdot 10 + 6.1 + 9.5 + 2 \cdot 8.6\}k = 8.8k$$

Same as in the previous Table

CAT Risk Pricing – Two Associated Issues

Issue 1: Addition, Attrition & Renewal

Q: When policies in the portfolio are dropped (added), the fair value of the remaining (existing) policies will change. Should pricing be updated promptly?

A: Strictly speaking, YES.

However, business practice may allow updates only at specific times, e.g., at renewal. Hence, pricing may be out-of-line between updates.

When policies are dropped (added), remaining (existing) policies are undercharged (overcharged).

CAT Risk Pricing – Two Associated Issues

Issue 2: Combination Explosion

Q: For a portfolio with many policies, the number of permutations of order increases exponentially. Wouldn't computations of the average risk load per Shapley Value be very time consuming?

A: YES. Although there are no theoretical difficulties, computation when portfolio has large number of policies may be cumbersome.

Suggest using Shapley (fair) pricing for portfolio with major contracts only.

For portfolio consisting mainly of small contracts, use following approximation algorithm instead…

CAT Risk Pricing – Portfolio with Many Accounts

- Risk load for Contract *m* in large portfolio

$$r_m = k\left\{0.2\sigma_m + 0.8\left(\sqrt{V} - \sqrt{V \setminus m}\right)\right\}$$

$$V = \text{Total variance of portfolio}$$

$$V \setminus m = \text{Variance without contract } m$$

$$\sigma_m = \text{sigma for contract } m$$

Derivation of Risk Load for Contract m in Large Portfolio

- The break-even charges for all accounts will determine the following relationship:

$$\sum_{m=1}^{n} r_m = k \sum_{m=1}^{n} [w\sigma_m + (1-w)(\sqrt{V} - \sqrt{V \setminus m})] = k\sqrt{V}$$

- From this, we develop the following general formula for the weights:

$$w = \frac{(1-n)\sqrt{V} + \sum_m \sqrt{V \setminus m}}{\sum_m [\sigma_m - \sqrt{V} + \sqrt{V \setminus m}]}$$

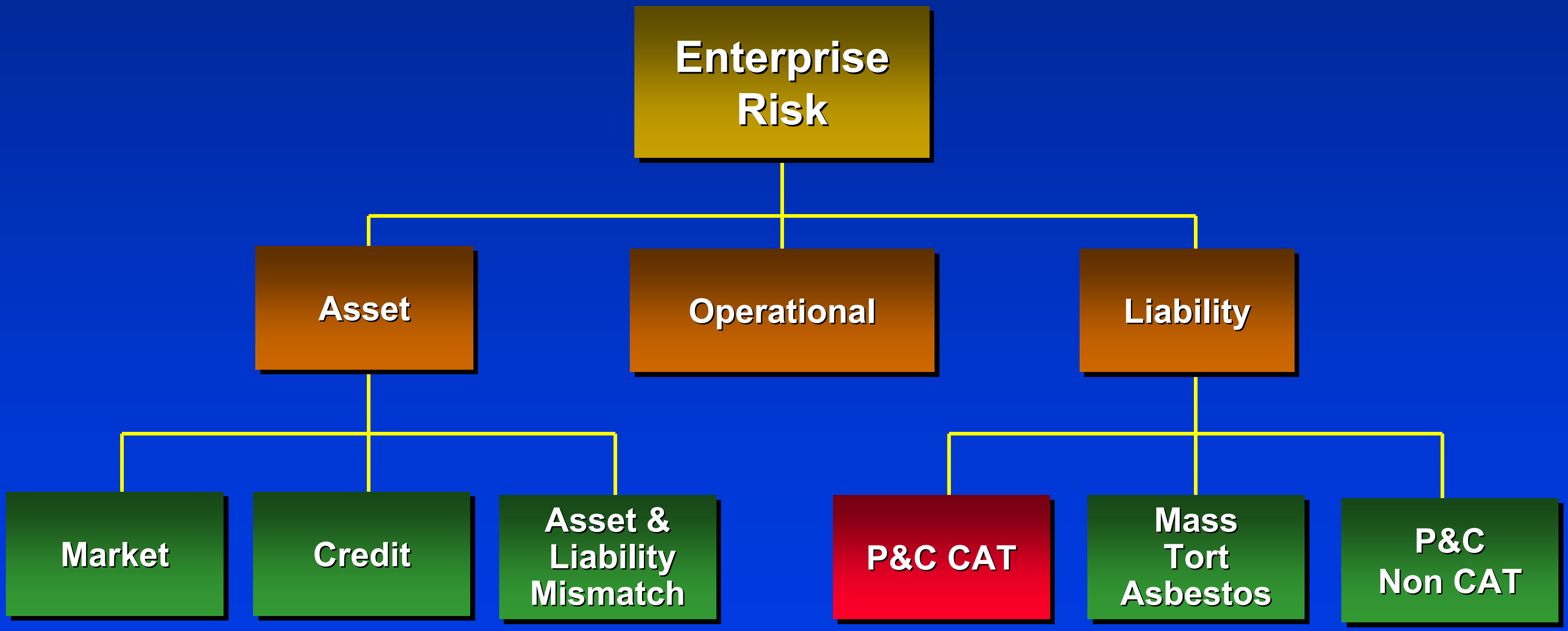
CAT Risk Pricing – Portfolio with Many Accounts

Weight based on different sizes of accounts, number of accounts, and correlation

n=	1	2	3	5	10	100
ρ=0	1	0.5	0.402	0.305	0.209	0.061
ρ=0.1	1	0.5	0.396	0.290	0.182	0.026
ρ=0.2	1	0.5	0.391	0.279	0.166	0.021
ρ=0.3	1	0.5	0.389	0.270	0.156	0.018
ρ=0.4	1	0.5	0.383	0.263	0.148	0.017
ρ=0.5	1	0.5	0.379	0.257	0.142	0.016
ρ=0.6	1	0.5	0.376	0.251	0.138	0.015
ρ=0.7	1	0.5	0.372	0.247	0.134	0.014
ρ=0.8	1	0.5	0.369	0.242	0.130	0.014
ρ=0.9	1	0.5	0.367	0.239	0.127	0.013

CAT Risk as Part of the Enterprise Profile

Apply Same Pricing Approach to Enterprise-Wide Risk



CAT Risk as Part of the Enterprise Profile

- CAT risk is only an element of the enterprise risk profile
- CAT risk is usually not strongly correlated with other risk elements
- Hence, it may provide diversification benefits in same way that uncorrelated CAT policies diversify a CAT-only portfolio

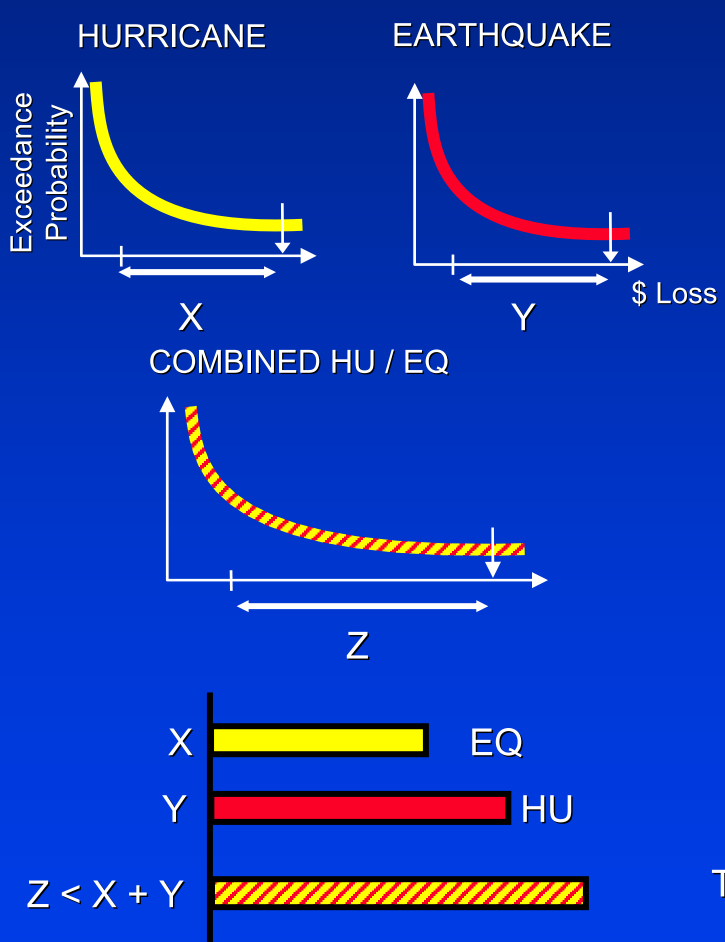
$$EC_{aggregate} < \sum_{i=1,n} EC_i$$

So, one more multiplier, called diversification factor d, is added to risk load parameter *k*, that is: $k = y \cdot d \cdot z$

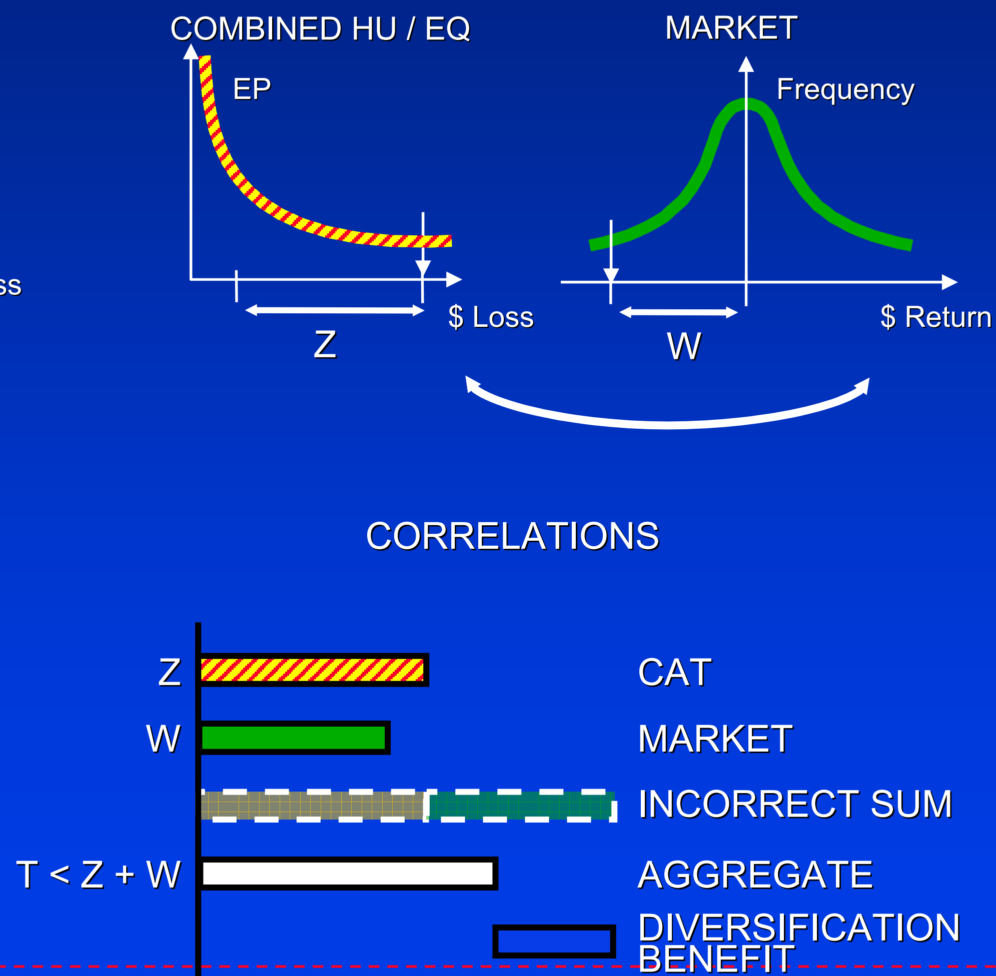
INTEGRATED FRAMEWORK ALLOWS CALCULATION OF DIVERSIFICATION BENEFITS

ILLUSTRATIVE

DIVERSIFICATION ACROSS PERILS



DIVERSIFICATION ACROSS RISK TYPES



CAT Risk as Part of the Enterprise Profile

Q: How much of the reduction in capital requirement should be attributed to each risk?

A: The Shapley Value can be applied to capital allocation for enterprise risks, analogous to its application to CAT-only portfolios.

CAT Risk as Part of the Enterprise Profile – the Shapley Value

- Paraphrasing previous example on CAT policies >> CAT portfolio, substitute category >> enterprise

Order of Addition	Category Risk Load, *ri*			Enterprise Risk Load
	C_1=*Credit*	C_2=*ALM*	C_3=*CAT*	
$\{C_1, C_2, C_3\}$	10*k*	5.5*k*	8.6*k*	$\sum_i r_i$ = 24.1*k*
$\{C_1, C_3, C_2\}$	10*k*	8*k*	6.1*k*	
$\{C_2, C_1, C_3\}$	5.5*k*	10*k*	8.6*k*	
$\{C_2, C_3, C_1\}$	4.6*k*	10*k*	9.5*k*	
$\{C_3, C_1, C_2\}$	6.1*k*	8*k*	10*k*	
$\{C_3, C_2, C_1\}$	4.6*k*	9.5*k*	10*k*	
Average	6.8*k*	8.5*k*	8.8*k*	

"Fair" Shapley Values

CAT Risk as Part of the Enterprise Profile – Diversification Benefit

Q: How much benefit can be derived from adding CAT risk to an enterprise profile?

A: Diversification benefit (reduction in capital requirement) similar to policy-portfolio aggregation will result; the benefit depends on the enterprise profile.

CAT Risk as Part of the Enterprise Profile – Diversification Benefit

- **Example:** Company A, CAT-heavy

	Stand Alone Cap	Contribution	Diversified Cap (Shapley Allocation)	Diversification Factor, *d*
Market+Alm	5,239	1.0%	587	11%
Credit	407	0.1%	44	11%
Liability	3,236	0.6%	28	1%
Catastrophe	503,822	97.5%	500,361	99%
Business	2,064	0.4%	19	1%
Event	1,722	0.3%	6	0%
Total Cap Req'd	516,489	100%	501,044	

Not much reduction in risk

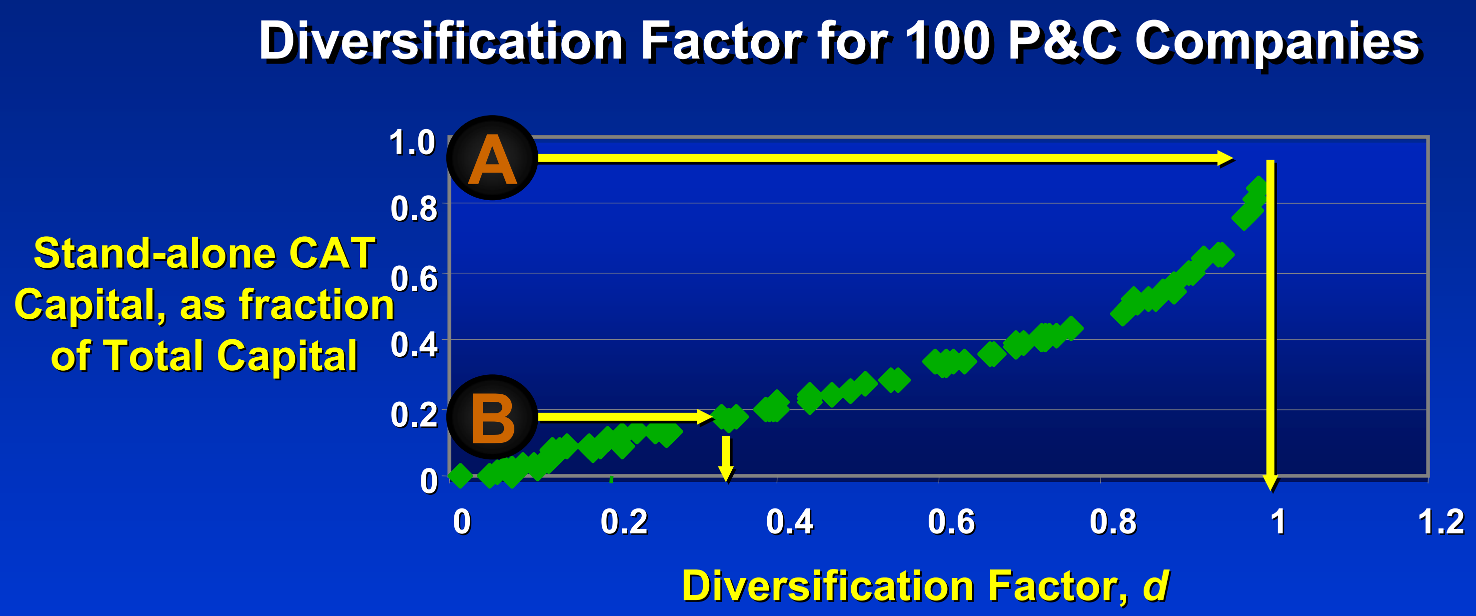
CAT Risk as Part of the Enterprise Profile – Diversification Benefit

- **Example:** Company B, CAT-light

	Stand Alone Cap	Contribution	Diversified Cap (Shapley Allocation)	Diversification Factor, *d*
Market+Alm	3,434,933	31.2%	2,540,282	74%
Credit	337,193	3.1%	171,949	51%
Liability	3,937,792	35.8%	2,873,225	73%
Catastrophe	1,378,597	12.5%	358,817	26%
Business	1,149,918	10.5%	652,030	57%
Event	757,553	6.9%	149,638	20%
Total Cap Req'd	10,995,987	100%	6,745,940	

Large reduction in risk

CAT Risk as Part of the Enterprise Profile – Diversification Benefit



- **Company A:** Companies with high CAT exposure will have almost no diversification benefit
- **Company B:** Companies with little or no CAT exposure will have significant diversification benefit for the CAT risk (include diversification factor *d* in pricing)

How to Select PML from Enterprise Risk Perspective

- Associated with the company's rating
- Based on CAT risk contribution

Summary

Risk Load = $k\Delta\sigma_{Shapley}$

Enterprise Risk Load (Shapley Allocation) (Diversification Factor)	Hurdle rate *y*	Distribution factor *z*	Diversification factor *d*
Portfolio Risk Load (Shapley Allocation)	Hurdle rate *y*	Distribution factor *z*	*
Policy Risk Load	Hurdle rate *y*	Distribution factor *z*	*

Ascending the CAT-Risk Pricing Tower

Questions

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